A unified approach to generate risk measures

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Abstract

In this paper we present a unified approach to derive many important classes of
premium principles, using the Markov inequality for tail probabilities. In addition, we
will recall some of the important characterization theorems of these risk measures.

Keywords: Insurance premium principle, Risk measure, Markov inequality

1 Introduction

In the economic and actuarial financial literature the concept of insurance premium prin-
ciples (risk measures) has been studied from different angles. In this paper we present a unified
approach to some important classes of premium principles, based on the Markov inequality
for tail probabilities. We prove that most well-known insurance premium principles can be
derived in this way. In addition, we will refer to some of the important characterization
theorems of these risk measures.

Basic material on utility theory and insurance goes back to Borch (1968, 1974), using
the utility concept of von Neumann and Morgenstern (1944). The fundamentals of premium
principles were laid by Bühlmann (1970) who introduced the zero-utility premium, Gerber
(1979) and comprehensively by Goovaerts \textit{et al.} (1984). Both the utility concept and the
mean-value premium principle, as well as the expected value principle, can be deduced from
certain axioms. An early source is Hardy \textit{et al.} (1952). The Swiss premium calculation
principle was introduced by Gerber (1974) and De Vijlder and Goovaerts (1979). A multi-
pllicative equivalent of the utility framework has led to the Orlicz principle as introduced

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by Haezendonck and Goovaerts (1982). A characterization for additive premiums has been introduced by Gerber and Goovaerts (1981), and led to the so-called mixture of Esscher premium principles. More recently, Wang (1996) introduced in the actuarial literature the distortion functions into the framework of risk measures, using Yaari’s (1987) dual theory of choice under risk. This approach also can be introduced in an axiomatic way. Artzner (1999) restricted the class of Orlicz premium principles by adding the requirement of translation invariance to its axioms, weakened by Jarrow (2002). This has mathematical consequences that are sometimes contrary to practical insurance applications. In the 1980’s the practical significance of the basic axioms has been discussed; see Goovaerts et al. (1984). On the same grounds Artzner (1999) provides an argumentation for selecting a set of desirable axioms. In Goovaerts et al. (2002) it is argued that there are no sets of axioms generally valid for all types of risky situations. There is a difference in desirable properties when one considers a risk measure for allocation of capital, a risk measure for regulating purposes or a risk measure for premiums. There is a parallel with mathematical statistics, where characteristics of distributions may have quite different meanings and uses, like e.g. the mean to measure central tendency, the variance to measure spread, the skewness to reflect asymmetry and the peakedness to measure the thickness of the tails. In an actuarial context, risk measures might have different properties than in other economic contexts. For instance, if we cannot assume that there are two different reinsurers willing to cover both halves of a risk separately, the risk measure (premium) for the entire risk should be larger than twice the original risk measure.

This paper aims to introduce a lot of the different risk measures (premium principles) now available, each with their desirable properties, within a unified framework based on the Markov inequality. To give an idea how this is achieved, we give a simple example.

**Example 1.1.** The exponential premium is derived as the solution to the utility equilibrium equation

\[
E[-e^{-\beta(w-X)}] = E[-e^{-\beta(w-\pi)}],
\]

where \(w\) is the initial capital and \(u(x) = -e^{-\beta x}\) is the utility attached to wealth level \(x\). This is equivalent to

\[
E[e^{-\beta(\pi-X)}] = 1,
\]

hence we get the explicit solution

\[
\pi = \frac{1}{\beta} \log E[e^{\beta X}].
\]

Applying \(\Pr[Y > y] \leq \frac{1}{y}E[Y]\) (Markov inequality) to \(Y = e^{\beta X}\) and \(y = e^{\beta \pi}\), we get the
following inequality for the survival probabilities with \( X \):

\[
\Pr[X > \pi] \leq \frac{1}{e^{\beta \pi}} \mathbb{E}[e^{\beta X}]. \tag{1.4}
\]

For the Markov bound to make sense, the r.h.s. of (1.4) must be at most 1. It equals 1 when \( \pi \) is equal to the exponential(\( \beta \)) premium with \( X \). This procedure leads to an equation which gives the premium for \( X \) from a Markov bound.

Two more things must be noted. First, for fixed \( \pi \), the risk aversion \( \beta_0 \) for which this bound for the tail probabilities is minimal is easily seen to satisfy \( \pi = \mathbb{E}[X e^{\beta_0 X}] / \mathbb{E}[e^{\beta_0 X}] \), which is the Esscher premium for \( X \) with parameter \( \beta_0 \). This way, also the Esscher premium has been linked to a Markov bound. The r.h.s. of (1.4) equals 1 for \( \beta = 0 \) as well, and is less than or equal to 1 for \( \beta \) in the interval \([0, \beta_1]\), where \( \beta_1 \) is the risk aversion for which the exponential premium equals \( \pi \). Second, if \( \pi = \frac{1}{\beta} \log \mathbb{E}[e^{\beta X}] \) holds, we have the following exponential upper bound for tail probabilities: for any \( k > 0 \),

\[
\Pr[X > \pi + k] \leq \frac{1}{e^{\beta(\pi+k)}} \mathbb{E}[e^{\beta X}] = e^{-\beta k}. \tag{1.5}
\]

Using variations of the Markov bound above, the various equations that generate various premium principles (or risk measures) can be derived. Section 2 presents a method to do this, Section 3 applies this method to many such principles, and discusses their axiomatic foundations as well as some other properties; Section 4 concludes.

## 2 Generating Markovian risk measures

Throughout this paper, for a random variable \( S \), we denote its cumulative distribution function (cdf) by \( F_S \). For any non-negative and non-decreasing function \( v(s) \) satisfying

\[
\mathbb{E}[v(S)] < +\infty, \tag{2.1}
\]

we define an associated random variable \( S^* \) such that it has a cdf with differential

\[
dF_{S^*}(s) = \frac{v(s)dF_{S}(s)}{\mathbb{E}[v(S)]}, \quad -\infty < s < +\infty. \tag{2.2}
\]

For the special case when \( v(s) = e^{hs} \) for some \( h \geq 0 \), we get the Esscher transform of \( dF_S(s) \). Since the differentials \( dF_{S^*}(s) \) and \( dF_S(s) \) cross exactly once, we have

\[
\Pr[S > \pi] \leq \Pr[S^* > \pi], \quad -\infty < \pi < +\infty. \tag{2.3}
\]

For any Lebesgue measurable bivariate function \( \phi(\cdot, \cdot) \) satisfying

\[
\phi(s, \pi) \geq \mathbb{I}_{(s > \pi)}, \tag{2.4}
\]

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we have the following inequalities:

\[ \Pr [S^* > \pi] = \mathbb{E} \left[ I_{[S^* > \pi]} \right] \leq \mathbb{E} [\phi(S^*, \pi)]. \]  (2.5)

Then it follows from (2.3) that

\[ \Pr [S > \pi] \leq \frac{\mathbb{E} [\phi(S, \pi)v(S)]}{\mathbb{E} [v(S)]}. \]  \[\text{[GMI]}\]

This is a generalized version of the Markov inequality, which has \( S \geq 0 \) with probability one and \( \pi \geq 0 \), \( \phi(s, \pi) = s/\pi \) and \( v(s) \equiv 1 \). Therefore, we denote it by the acronym [GMI]. Similar discussions can be found in Runnenburg and Goovaerts (1985), where the functions \( v(\cdot) \), \( \phi(\cdot, \cdot) \) are specially chosen as \( v(\cdot) \equiv 1 \), respectively \( \phi(s, \pi) = f(s)/f(\pi) \) for some non-negative and non-decreasing function \( f(\cdot) \).

For the inequality [GMI] to make sense, the bivariate function \( \phi(s, \pi) \) given in (2.4) and the random variable \( S \) should satisfy

\[ \mathbb{E} [\phi(S, \pi)v(S)] < +\infty \]  (2.6)

for all relevant \( \pi \). Note that if (2.6) holds for some \(-\infty < \pi < +\infty\) then (2.1) does. By assuming (2.6), it is clear that the family of random variables \( S \) considered in the inequality [GMI] is restricted, i.e. the right tail of \( S \) can’t be arbitrarily heavy. For the given functions \( \phi(\cdot, \cdot) \) and \( v(\cdot) \) as above, we introduce below a family of all admissible random variables which satisfy (2.6):

\[ S_{\phi, v} = \{ S : \mathbb{E} [\phi(S, \pi)v(S)] < +\infty \text{ for all large } \pi \}. \]  (2.7)

Sometimes we are interested in the case that there exists a minimal value \( \pi_M^{(\alpha)} \) such that [GMI] gives a bound

\[ \Pr [S > \pi_M] \leq \frac{\mathbb{E} [\phi(S, \pi_M)v(S)]}{\mathbb{E} [v(S)]} \leq \alpha \leq 1. \]  (2.8)

For each \( 0 \leq \alpha \leq 1 \), the restriction (2.4) on \( \phi(\cdot, \cdot) \) allows us further to introduce a subfamily of \( S_{\phi, v} \) as below:

\[ S_{\phi, v, \alpha} = \left\{ S : \frac{\mathbb{E} [\phi(S, \pi_M)v(S)]}{\mathbb{E} [v(S)]} \leq \alpha \text{ for all large } \pi \right\}. \]  (2.9)

If in (2.4) the function \( \phi(s, \pi) \) is strictly smaller than 1 for at least one point \((s, \pi)\), then it is not difficult to prove that there are some values of \( 0 \leq \alpha < 1 \) such that the subfamilies \( S_{\phi, v, \alpha} \) are not empty. We also note that \( S_{\phi, v, \alpha} \) increases in \( \alpha \geq 0 \).

Hereafter, for a real function \( f(\cdot) \) defined on an interval \( D \) and a constant \( b \) in the range of the function \( f \), we write an equation \( f(\pi) = b \) with understanding that its root is the
minimal value of $\pi$ satisfying $f(\pi) \leq b$ and $\max\{f(x)|x \in (\pi - \varepsilon, \pi + \varepsilon) \cap D\} \geq b$ for any $\varepsilon > 0$. With this convention, the minimal value $\pi^{(\alpha)}_M$ such that the second inequality in (2.8) holds is simply the solution of the equation

$$\frac{\mathbb{E}[\phi(S, \pi_M)v(S)]}{\mathbb{E}[v(S)]} = \alpha.$$  \hspace{1cm} [UE_{\alpha}]

When $\alpha = 1$, we call

$$\frac{\mathbb{E}[\phi(S, \pi_M)v(S)]}{\mathbb{E}[v(S)]} = 1$$  \hspace{1cm} [UE]

as the unifying equation, or [UE] in acronym. This equation will act as the unifying form to generate many well-known risk measures. The equation [UE] gives the minimal percentile for which the upper bound for the tail probability of $S$ still makes sense. It will turn out that these minimal percentiles correspond to several well-known premium principles (risk measures). It is clear that the solution of the equation [UE] is not smaller than the minimal value of the random variable $S$.

**Definition 2.1.** Let $S$ be a random variable from the family $S_{\phi,v,\alpha}$ for some $0 \leq \alpha \leq 1$. The solution $\pi^{(\alpha)}_M$ of the equation $[\text{UE}_\alpha]$ is called a *Markovian risk measure* of the random variable $S$ at level $\alpha$.

**Remark 2.2.** A Markovian risk measure provides an upper bound for the VaR at the same level. By selecting appropriate functions $\phi$ the Markovian risk measures can reflect desirable properties when adding random variables in addition to their dependence structure.

**Example 2.3.** Let $X_1$ and $X_2$ be two random variables, with Markovian risk measures $\pi^{(\alpha)}_M(X_1)$ and $\pi^{(\alpha)}_M(X_2)$. Then we have

$$\Pr\left[X_1 > \pi^{(\alpha)}_M(X_1)\right] \leq \alpha, \quad \Pr\left[X_2 > \pi^{(\alpha)}_M(X_2)\right] \leq \alpha.$$  \hspace{1cm} (2.10)

We can obtain from the equation $[\text{UE}_\alpha]$ that

$$\Pr\left[X_1 + X_2 > \pi^{(\alpha)}_M(X_1) + \pi^{(\alpha)}_M(X_2)\right] \leq \alpha$$  \hspace{1cm} (2.11)

1) in case $X_1$ and $X_2$ are independent when the risk measure $\pi^{(\alpha)}_M$ involved is additive for sums of independent risks;
2) in case $X_1$ and $X_2$ are comonotonic when the risk measure $\pi^{(\alpha)}_M$ involved is additive for sums of comonotonic risks;
3) for any $X_1$ and $X_2$, regardless of their dependence structure, in case a subadditive risk measure $\pi^{(\alpha)}_M$ is applied.
3 Some Markovian risk measures

In what follows we will provide a list of some important insurance premium principles (or risk measures) and show how they can be derived from the equation [UE]. We will also list a set of basic underlying axioms. In practice, for different situations different sets of axioms are needed.

3.1 The mean value principle

The mean value principle has been characterized by Hardy et al. (1952); see also Goovaerts et al. (1984), Chapter 2.8, in the framework of insurance premiums.

Definition 3.1. Let $S$ be a risk variable. The mean value risk measure is obtained as the root of the equation $f(\pi) = E[f(S)]$, where $f$ is a non-decreasing and non-negative function such that $E[f(S)]$ converges.

Clearly, we can obtain the mean value risk measure by choosing in the equation [UE] the functions $\varphi(s, \pi) = f(s)/f(\pi)$ and $v(\cdot) \equiv 1$. As verified in Goovaerts et al. (1984), p. 57-61, this principle can be characterized by the following axioms (necessary and sufficient conditions):

A1.1. $\pi(c) = c$ for any degenerative risk $c$;

A1.2. $\Pr[X \leq Y] = 1 \implies \pi(X) \leq \pi(Y)$;

A1.3. If $\pi(X) = \pi(X')$, $Y$ is a random variable and $I$ is a Bernoulli variable independent of $\{X, X', Y\}$, then $\pi(IX + (1 - I)Y) = \pi(IX' + (1 - I)Y)$.

Remark 3.2. This last axiom can be expressed in terms of distribution functions by assuming that mixing $F_Y$ with $F_X$ or with $F_{X'}$ leads to the same risk measure, as long as the mixing weights are the same.

Under the condition that $E[f(S)]$ converges one obtains as an upper bound for the survival probability

$$\Pr[S > \pi + u] \leq \frac{E[f(S)]}{f(\pi + u)} = \frac{f(\pi)}{f(\pi + u)}. \quad (3.1)$$

Especially when $E[e^{\lambda S}] < \infty$ for some $\lambda > 0$, one obtains $e^{-\lambda u}$ as an upper bound for the probability $\Pr[S > \pi + u]$, see the example in Section 1. In case $\pi_\alpha$ is the root of $E[f(S)] = \alpha f(\pi)$, by the inequality [GMI] one gets $\Pr[S > \pi_\alpha] \leq \alpha$. 

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3.2 The zero-utility premium principle

The zero-utility premium principle was introduced by Bühlmann (1970).

**Definition 3.3.** Let \( u \) be a non-decreasing utility function. The zero-utility premium \( \pi(S) \) is the solution of \( u(0) = E[u(\pi - S)] \).

For simplicity we assume that the risk or the utility function is bounded from above. Because \( u(\cdot) \) and \( u(\cdot) + c \) define the same ordering in expected utility, the utility is determined such that \( u(x) \to 0 \) as \( x \to +\infty \). To obtain the zero-utility premium principle, one chooses in the equation [UE] the functions \( \phi(s, \pi) = u(\pi - s)/u(0) \) and \( v(\cdot) \equiv 1 \).

In order to relate the utility to the VaR one should proceed as follows. By the inequality [GMI], we get

\[
\Pr [S > \pi_{\alpha}] \leq E \left[ \frac{-u(\pi_{\alpha} - S)}{-u(0)} \right] = \alpha, \tag{3.2}
\]

where \( \pi_{\alpha} \) is the solution of the equation \( E[u(\pi - S)] = \alpha u(0) \). The result obtained here requires that the utility function \( u(\cdot) \) is bounded from below. However, this restriction can be weakened by considering limit for translated utility functions.

Let the symbol \( \preceq_{eu} \) represent the weak order with respect to the zero-utility premium principle, that is, \( X \preceq_{eu} Y \) means that \( X \) is preferable to \( Y \). We write \( X \sim_{eu} Y \) if both \( X \preceq_{eu} Y \) and \( Y \preceq_{eu} X \). It is well-known that the preferences of a decision maker between risks can be described by means of comparing expected utility as a measure of the risk if they fulfill the following five axioms which are due to von Neumann and Morgenstern (1944) (combining Denuit et al. (1999) and Wang and Young (1998)):

A2.1. If \( F_X = F_Y \) then \( X \sim_{eu} Y \);

A2.2. The order \( \preceq_{eu} \) is reflexive, transitive and complete;

A2.3. If \( X_n \preceq_{eu} Y \) and \( F_{X_n} \to F_X \) then \( X \preceq_{eu} Y \);

A2.4. If \( F_X \geq F_Y \) then \( X \preceq_{eu} Y \);

A2.5. If \( X \preceq_{eu} Y \) and if the distribution functions of \( X'_p \) and \( Y'_p \) are given by \( F_{X'_p}(x) = pF_X(x) + (1-p)F_Z(x) \) and \( F_{Y'_p}(x) = pF_Y(x) + (1-p)F_Z(x) \), where \( F_Z \) is an arbitrary distribution function, then \( X'_p \preceq_{eu} Y'_p \) for any \( p \in [0,1] \).

From these axioms, the existence of a utility function \( u(\cdot) \) can be proven, with the property that \( X \preceq_{eu} Y \) if and only if \( E[u(-X)] = E[u(-Y)] \).
3.3 The Swiss premium calculation principle

The Swiss premium principle was introduced by Gerber (1974) to put the mean value principle and the zero-utility principle in the same framework.

Definition 3.5. Let \( w(\cdot) \) be a non-negative and non-decreasing function on \( \mathbb{R} \) and \( 0 \leq z \leq 1 \) be a parameter. Then the Swiss premium principle \( \pi = \pi(S) \) is the root of the equation

\[
E[w(S - z\pi)] = w((1 - z)\pi).
\] (3.3)

This equation is the special case of [UE] with \( \phi(s, \pi) = w(s - z\pi)/w((1 - z)\pi) \) and \( v(\cdot) \equiv 1 \). It is clear that \( z = 0 \) provides us with the mean value premium, while \( z = 1 \) gives the zero-utility premium. Recall that by the inequality [GMI], the root \( \pi_\alpha \) of the equation

\[
E[w(S - z\pi)] = \alpha w((1 - z)\pi)
\]

determines an upper bound for the VaR \( \alpha \).

Remark 3.6. Because one still may choose \( w \), it can be arranged to have supplementary properties for the risk measures. Indeed if we assume that \( w(\cdot) \) is convex, we have

\[
X \leq_{cx} Y \implies \pi(X) \leq \pi(Y).
\] (3.4)

See e.g. Dhaene et al. (2002a,b) for the definition of the convex order. For two random pairs \((S_1, S_2)\) and \((\tilde{S}_1, \tilde{S}_2)\), we call \((S_1, S_2)\) more related than \((\tilde{S}_1, \tilde{S}_2)\) if the probability \( \Pr[S_1 \leq x, S_2 \leq y] \) that \( S_1 \) and \( S_2 \) are both small is larger than this probability for \( \tilde{S}_1 \) and \( \tilde{S}_2 \), for all \( x \) and \( y \); see e.g. Kaas et al. (2001), Chapter 10.6. In this case one gets

\[
\pi(\tilde{S}_1 + \tilde{S}_2) \leq \pi(S_1 + S_2).
\] (3.5)

Hence, the risk measure of the sum of a pair of r.v.’s with the same marginal distributions depends on the dependence structure.

Gerber (1974) proves the following characterization:

Remark 3.7. Let \( w \) be strictly increasing and continuous, then the Swiss premium calculation principle generated by \( w \) is additive for independent risks if and only if \( w \) is exponential or linear.

3.4 The Orlicz premium principle

The Orlicz principle was introduced by Haezendonck and Goovaerts (1982) as a multiplicative equivalent of the zero-utility principle. To introduce this premium principle, they used the concept of a Young function \( \psi \), which is a mapping from \( \mathbb{R}_0^+ \) into \( \mathbb{R}_0^+ \) that can be written as an integral of the form

\[
\psi(x) = \int_0^x f(t)dt, \quad x \geq 0,
\] (3.6)
where \( f \) is a left-continuous, increasing function on \( \mathbb{R}_0^+ \) with \( f(0) = 0 \) and \( \lim_{x \to +\infty} f(x) = +\infty \). It is seen that a Young function \( \psi \) is absolutely continuous, convex and strictly increasing, and has \( \psi'(0) = 0 \). We say that \( \psi \) is normalized if \( \psi(1) = 1 \).

**Definition 3.8.** Let \( \psi \) be a normalized Young function. The root of the equation

\[
E[\psi(S/\pi)] = 1
\]

is called the *Orlicz premium principle* of the risk \( S \).

The unified approach follows from the equation [UE] with \( \phi(s, \pi) \) replaced by \( \psi(s/\pi) \) and \( v(s) \equiv 1 \). The Orlicz premium satisfies the following properties:

A4.1. \( \Pr[X \leq Y] = 1 \implies \pi(X) \leq \pi(Y) \);

A4.2. \( \pi(X) = 1 \) when \( X \equiv 1 \);

A4.3. \( \pi(aX) = a\pi(X) \) for \( a > 0 \) and any risk \( X \);

A4.4. \( \pi(X + Y) \leq \pi(X) + \pi(Y) \).

**Remark 3.9.** A4.3 above says that the Orlicz premium principle is positively homogenous. In the literature, positive homogeneity is often confused with currency independence. As an example, we look at the standard deviation principle \( \pi_1(X) = E[X] + \alpha \cdot \sigma[X] \) and the variance principle \( \pi_2(X) = E[X] + \beta \cdot \text{Var}[X] \), where \( \alpha \) and \( \beta \) are two positive constants, \( \alpha \) is dimension-free but the dimension of \( 1/\beta \) is money. Clearly \( \pi_1(X) \) is positive homogenous but \( \pi_2(X) \) is not. But it stands to reason that when applying a premium principle, if the currency is changed, so should all constants having dimension money. So going from BFr to Euro, where 1 Euro \( \approx 40 \) BFr, the value of \( \beta \) in \( \pi_2(X) \) should be adjusted by the same factor. In this way both \( \pi_1(X) \) and \( \pi_2(X) \) are independent of the monetary unit.

**Remark 3.10.** These properties remain exactly the same for risks that may also be negative, as the one that are called coherent by Artzner (1999). Indeed if \( \pi(-1) = -1 \) one extends these properties to random variables supported on the whole line \( \mathbb{R} \), then

\[
\pi(X + a - a) \leq \pi(X + a) - a.
\]

Hence \( \pi(X + a) \geq \pi(X) + a \) and consequently \( \pi(X + a) = \pi(X) + a \).

The interested reader is referred to Haezendonck and Goovaerts (1982). If in addition translation invariance is imposed for non-negative risks, it turns out that the only coherent risk measure for non-negative risks within the class of Orlicz principles is an expectation \( \pi(X) = E[X] \).
Remark 3.11. The Orlicz principle can also be generalized to cope with \( \text{VaR}_\alpha \). Actually, from the inequality [GMI], the solution \( \pi_\alpha \) of the equation \( \mathbb{E}[\psi(S/\pi)] = \alpha \) gives \( \Pr[S > \pi_\alpha] \leq \alpha \).

3.5 More general risk measures derived from Markov bounds

For this section, let us confine to distributions with the same expectation. We consider more general risk measures derived from Markov bounds, applied to sums of pairs of random variables, which may or may not be independent. The generalization consists in the fact that we consider the dependence structure to some extent in the risk premium, letting the premium for the sum \( X + Y \) depend both on the distribution of the sum \( X + Y \) and on the distribution of the sum \( X^c + Y^c \) of the comonotonic (maximally dependent) copies of the r.v.’s \( X \) and \( Y \). Because of this, we would rather denote the premium for the sum \( X + Y \) by \( \pi(X,Y) \) rather than by \( \pi(X + Y) \). When the r.v.’s \( X \) and \( Y \) are comonotonic, however, there’s no any difference in understanding between the two symbols \( \pi(X,Y) \) and \( \pi(X + Y) \).

Taking \( \pi(X) \) simply equal to \( \pi(X,0) \), we consider the following properties:

A5.1. \( \pi(aX) = a\pi(X) \) for any \( a > 0 \);

A5.2. \( \pi(X + b) = \pi(X) + b \) for any \( b \in \mathbb{R} \);

A5.3. \( \pi(X,Y) \leq \pi(X) + \pi(Y) \);

A5.3. \( \pi(X,Y) \geq \pi(X) + \pi(Y) \);

A5.3. \( \pi(X,Y) = \pi(X) + \pi(Y) \).

We remark that subadditivity A5.3.a is only realistic in case diversification of risks is possible. However, this is rarely the case in insurance. Subadditivity gives rise to easy mathematics because distance functions can be used. The mathematics for the superadditive case is much harder.

Let \( \psi \) be an increasing and convex function on \( \mathbb{R} \), satisfying \( \lim_{x \to +\infty} \psi(x) = +\infty \). We get, by choosing \( v(\cdot) = 1 \),

\[
\phi(X,Y,\pi) = \frac{1}{\psi(1)} \cdot \psi \left( \frac{(X + Y - F_{X^c + Y^c}(p))_+}{\pi - F_{X^c + Y^c}(p)} \right) \tag{3.9}
\]

and by solving [UE] for \( \pi \), the following risk measure for the sum of two random variables:

\[
\mathbb{E} \left[ \psi \left( \frac{(X + Y - F_{X^c + Y^c}(p))_+}{\pi(X,Y) - F_{X^c + Y^c}(p)} \right) \right] = \psi(1) \tag{3.10}
\]
for some parameter $0 < p < 1$. Hereafter, the $p$th quantile of a random variable $X$ with d.f. $F_X$ is, as usual, defined by

$$F_X^{-1}(p) = \inf \{ x \in \mathbb{R} | F_X(x) \geq p \}, \quad p \in [0, 1].$$

(3.11)

It is easily seen that there exists a unique constant $a(p) > 0$ such that

$$E \left[ \psi \left( \frac{(X + Y - F_X^{-1}(p) - F_Y^{-1}(p))}{a(p)} \right) \right] = \psi(1).$$

(3.12)

Thus $\pi(X, Y) = F_X^{-1}(p) + a(p)$. Especially, letting $Y$ be degenerate at 0 we get $\pi(X) > F_X^{-1}(p)$.

Now we check that A5.1, A5.2 and A5.3 are satisfied by $\pi$ (subadditive case). In fact, the proofs for the first two axioms are trivial. As for A5.3, we derive

$$E \left[ \psi \left( \frac{(X + Y - F_X^{-1}(p) - F_Y^{-1}(p))}{\pi(X) + \pi(Y) - F_X^{-1}(p) - F_Y^{-1}(p)} \right) \right] \leq \psi \left( \frac{\pi(X) - F_X^{-1}(p)}{\pi(X) + \pi(Y) - F_X^{-1}(p) - F_Y^{-1}(p)} \cdot \frac{X - F_X^{-1}(p)}{\pi(X) - F_X^{-1}(p)} \right) + \psi \left( \frac{\pi(Y) - F_Y^{-1}(p)}{\pi(Y) - F_Y^{-1}(p)} \cdot \frac{Y - F_Y^{-1}(p)}{\pi(Y) - F_Y^{-1}(p)} \right)

\leq \psi(1) \quad \text{by A5.3.}

(3.13)

This proves A5.3.

**Remark 3.12.** If the function $\psi(\cdot)$ above is restricted to satisfy $\psi(1) = \psi'(1)$, then it can be proven that the risk measure

$$\pi_l(X) = F_X^{-1}(p) + \mathbb{E} \left[ (X - F_X^{-1}(p))_+ \right]$$

(3.14)

gives the lowest generalized Orlicz measure. In fact, since $\psi$ is convex on $\mathbb{R}$ and satisfies $\psi(1) = \psi'(1)$, we have

$$\psi((x)_+) \geq \psi(1) \cdot (x)_+ \quad \text{for any } x \in \mathbb{R}.$$
Let \( \pi(X) \) be a generalized Orlicz risk measure of the risk \( X \), that is, \( \pi(X) \) is the solution of the equation

\[
E \left[ \psi \left( \frac{(X - F_X^{-1}(p))_+}{\pi(X) - F_X^{-1}(p)} \right) \right] = \psi(1). \tag{3.16}
\]

By (3.15) and recalling that \( \pi(X) > F_X^{-1}(p) \), we have

\[
\psi(1) = E \left[ \psi \left( \frac{(X - F_X^{-1}(p))_+}{\pi(X) - F_X^{-1}(p)} \right) \right] \geq \psi(1) \cdot E \left[ \frac{(X - F_X^{-1}(p))_+}{\pi(X) - F_X^{-1}(p)} \right], \tag{3.17}
\]

which implies that

\[
\pi(X) \geq F_X^{-1}(p) + E \left[ (X - F_X^{-1}(p))_+ \right] = \pi_l(X). \tag{3.18}
\]

**Remark 3.13.** Now we consider the risk measure

\[
E \left[ \psi \left( \frac{(X - F_X^{-1}(p))_+}{\pi(X) - F_X^{-1}(p)} \right) \right] = 1 - p \tag{3.19}
\]

for some parameter \( 0 < p < 1 \). Similarly as in Remark 3.12, if the function \( \psi(\cdot) \) is restricted to satisfy \( \psi(1) = \psi'(1) \), we obtain the lowest risk measure as

\[
\pi(X) = F_X^{-1}(p) + \frac{1}{1 - p} E \left[ (X - F_X^{-1}(p))_+ \right] = E \left[ X \mid X > F_X^{-1}(p) \right]. \tag{3.20}
\]

**Remark 3.14.** Another choice is to consider the root of the equation

\[
E \left[ \frac{1}{\psi(1)} \cdot \psi \left( \frac{|X - E[X]|}{\pi(X) - E[X]} \right) \right] = \alpha, \tag{3.21}
\]

defining, in general terms, a risk measure for the deviation from the expectation. As a special case when \( \psi(t) \equiv t^2 I_{t\geq0} \) one gets

\[
\pi(X) = E[X] + \frac{\sigma |X|}{\sqrt{\alpha}}. \tag{3.22}
\]

**Remark 3.15.** One could consider a risk measure measuring central tendency \( \pi_c(X) \) which is additive, and define an “error” \( \rho(\cdot) \) by considering

\[
E \left[ \psi \left( \frac{|X - \pi_c(X)|}{\rho(X) - \pi_c(X)} \right) \right] = \psi(1). \tag{3.23}
\]

If \( \pi_c(X) \) is partially homogenous, translation invariant and additive, then \( \rho(X) \) is partially homogenous and translation invariant. The measure \( \rho(X) \) maybe subadditive or superadditive, depending on the convexity or concavity of the function \( \psi(\cdot) \).
3.6 Yaari’s dual theory of choice under risk

Yaari (1987) introduced the dual theory of choice under risk. It was used by Wang (1996), who introduced distortion functions in the actuarial literature. A distortion function is defined as a non-decreasing function $g : [0, 1] \rightarrow [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$.

**Definition 3.16.** Let $S$ be a risk variable with a d.f. $F_S$, and $g(\cdot)$ be a distortion function defined above, then the distortion risk measure associated with the distortion function $g$ is defined by

$$
\pi = -\int_{-\infty}^{0} [1 - g(1 - F_S(x))] \, dx + \int_{0}^{+\infty} g(1 - F_S(x)) \, dx.
$$

Choosing the function $\phi(\cdot, \cdot)$ in the equation [UE] such that $\phi(s, \pi) = |s|/\pi$ and using the left-hand derivative $g'(1 - F_S(s))$ instead of $v(s)$, then by integration by parts we get the desired unifying approach.

This risk measure can be characterized by the following axioms:

A6.1. $\Pr[X \leq Y] = 1 \implies \pi(X) \leq \pi(Y)$;

A6.2. If risks $X$ and $Y$ are comonotonic then $\pi(X + Y) = \pi(X) + \pi(Y)$;

A6.3. $\pi(1) = 1$.

**Remark 3.17.** It is clear that this principle results in large upper bounds because

$$
\Pr[X \geq \pi + u] \leq \mathbb{E} \left[ \frac{X \cdot g'(1 - F_X(X))}{\pi + u} \right] = \frac{\pi}{\pi + u}.
$$

It is also clear that the set of risks for which $\pi$ is finite contains all risks with finite expectations.

3.7 A mixture of Esscher principles

The mixture of Esscher premium was introduced by Gerber and Goovaerts (1981). It is defined as follows:

**Definition 3.18.** Let $S$ be a bounded random variable. A principle $\pi = \pi(S)$ is said to be a mixture of Esscher principles if it is of the form

$$
\pi_F(S) = F(-\infty)\phi(-\infty) + \int_{-\infty}^{+\infty} \phi(t) \, dF(t) + (1 - F(+\infty))\phi(+\infty),
$$

where $F$ is a non-decreasing function satisfying $0 \leq F(t) \leq 1$ and $\phi$ is of the form

$$
\phi(t) = \phi_S(t) = \frac{d}{dt} \log \mathbb{E}[e^{tS}] , \quad t \in \mathbb{R}.
$$
Actually we can regard $F$ as a possible defective cdf with mass at both $-\infty$ and $+\infty$. Since the variable $S$ is bounded, $\phi(-\infty) = \min[S]$ and $\phi(+\infty) = \max[S]$. In addition, $\phi_S(t)$ is the Esscher premium of $S$ with parameter $t \in \mathbb{R}$.

In the special case where the function $F$ is defined on the interval $[0, \infty]$, the mixture of Esscher principles is a mixture of premiums with a non-negative safety loading coefficient. We show that in this case the mixture of Esscher premium can also be derived from the Markov inequality. Actually,

$$\pi_F(S) = \int_0^{+\infty} \phi(t)dF(t) + (1 - F(+\infty))\phi(+\infty) = \int_{[0, +\infty]} \phi(t)dF(t).$$

(3.28)

It can be shown that the mixture of Esscher principle is translation invariant. Hence in what follows, without loss of generality we simply assume that $\min[S] \geq 0$ because otherwise a translation on $S$ can be used. We notice that, for any $t \in [0, +\infty]$,

$$\phi(t) = \frac{E[Se^{tS}]}{E[e^{tS}]} \geq E[S].$$

(3.29)

The inequality (3.29) can, for instance, be deduced from the fact that the variables $S$ and $e^{tS}$ are comonotonic, hence positively correlated. Since we have assumed that $\min[S] \geq 0$, now we choose in [GMI] the functions $v(\cdot) \equiv 1$ and $\phi(s, \pi) = s/\pi$, then obtain that

$$\Pr[S > \pi + u] \leq \frac{1}{\pi} E[S] \leq \frac{1}{\pi} \int_{[0, +\infty]} \phi(t)dF(t),$$

(3.30)

where, the last step in (3.30) is due to the inequality (3.29) and the fact that $F([0, +\infty]) = 1$. Letting the r.h.s. of (3.30) be equal to 1, we immediately obtain (3.28).

We now verify another result: the tail probability $\Pr[S > \pi + u]$ decreases exponentially fast in $u \in [0, +\infty)$. The proof is not difficult. Actually, since the risk variable $S$ is bounded, it holds for any $\alpha > 0$ that

$$\Pr[S > \pi + u] \leq \exp\{-\alpha(\pi + u)\} \cdot E[\exp\{\alpha S\}].$$

(3.31)

Hence, in order to get the announced result, it suffices to prove that, for some $\alpha > 0$,

$$E[\exp\{\alpha S\}] \leq \exp\{\alpha \pi\} = \exp\left\{\alpha \int_{[0, +\infty]} \phi(t)dF(t)\right\},$$

or equivalently to prove that, for some $\alpha > 0$,

$$\log E[\exp\{\alpha S\}] \leq \alpha \int_{[0, +\infty]} \phi(t)dF(t).$$

(3.32)
In the trivial case when the risk $S$ is degenerate, both sides of (3.32) are equal for any $\alpha > 0$. In the other case, noticing that $F$ is not degenerate and that the Esscher premium $\phi(t)$ is strictly increasing in $t \in [0, +\infty]$ we can find some $\alpha_0 > 0$ such that

$$\phi(\alpha) \leq \int_{[0, +\infty]} \phi(t)dF(t) \quad (3.33)$$

holds for any $\alpha \in [0, \alpha_0]$. Thus in any case we obtain that (3.32) holds for any $\alpha \in [0, \alpha_0]$.

We summarize:

**Remark 3.19.** For the mixture of Esscher premiums $\pi$ defined above, if $F$ is concentrated on $[0, +\infty]$, then

$$\Pr [S > \pi + u] \leq \exp\{-\alpha_0 u\} \quad (3.34)$$

holds for any $u \geq 0$, where the constant $\alpha_0 > 0$ is the solution of the equation $\phi(\alpha) = \int_{[0, +\infty]} \phi(t)dF(t)$.

The mixture of Esscher premiums is characterized by the following axioms; see Gerber and Goovaerts (1981):

A7.1. $\phi_{X_1}(t) \leq \phi_{X_2}(t) \quad \forall t \in \mathbb{R} \implies \pi_F(X_1) \leq \pi_F(X_2)$;

A7.2. It holds for any two independent risks $X_1$ and $X_2$ that $\pi_F(X_1 + X_2) = \pi_F(X_1) + \pi_F(X_2)$.

Hence this risk measure is additive for independent risks. When the function $F$ in (3.28) is defined on the interval $[0, \infty]$, the premium contains a positive safety loading.

4 Conclusions

This note shows how many of the usual premium calculation principles (or risk measures) can be deduced from a generalized Markov inequality. All risk measures provide information concerning the VaR, as well as the asymptotic behavior of $\Pr [S > \pi + u]$. Therefore, the effect of using a risk measure and requiring additional properties is equivalent to making a selection of admissible risks. Notice that when using a risk measure, additional requirements are usually needed about convergence of certain integrals. In this way, the set of admissible risks is restricted, e.g. the one having finite mean, finite variance, finite moment generating function and so on.

References


